

Effective Matter Cosmologies of Massive Gravity: Physical Fluids

Nejat Tevfik Yılmaz

Department of Electrical and Electronics Engineering,

Yaşar University,

Selçuk Yaşar Kampüsü

Üniversite Caddesi, No:35-37,

AğaçlıYol, 35100,

Bornova, İzmir, Turkey.

`nejat.yilmaz@yasar.edu.tr`

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Abstract

We derive new cosmological solutions of the ghost-free massive gravity with a general background metric in which the contribution of the mass sector to the metric one is modeled by an effective cosmological constant and an ideal fluid which obeys the first law of thermodynamics; thus it satisfies the ordinary energy-momentum conservation or continuity equation.

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1 Introduction

In the following work we construct a new class of Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmological solutions of the ghost-free [1, 2] massive gravity theory [3, 4, 5, 6, 7] which is a nonlinearized version of the

Fierz-Pauli [8] massive gravity. The cosmological solutions of the ghost-free massive gravity have been extensively studied in recent years. It has already been shown that for a flat background metric there exist open FLRW cosmological solutions [9] but no flat or closed ones [10]. Hence the program has evolved to search the cosmological solutions for the de Sitter [11] and the FLRW type [12] background metrics. The underlying motivation to construct the cosmological solutions of ghost-free massive gravity in these as well as other works [13, 14, 15, 16, 17, 18, 19] is to obtain phenomenologically acceptable self-accelerating solutions which will admit self-acceleration without a need for a matter-generated cosmological constant. However, the solutions for the above-mentioned natural background metrics have been shown to possess stability problems [12, 20, 21] which turn them into physically unacceptable candidates. One way to overcome the stability problem in the cosmological solutions is to give up the homogeneity and, or the isotropy of the physical metric. In that case one may recover the FLRW cosmologies at the regime of the Compton wavelength of the mass if it is of the order of the Hubble constant [22]. Another route is to consider nonstandard background metrics for which the stability and symmetry considerations of the cosmological solutions are still the primary concern for the physical qualification of the solution.

In the general literature of deriving cosmological solutions of massive gravity there exist two technical approaches. One of them is to fix the background metric as a flat one and to solve the field equations upon introducing an ansatz for the Stückelberg fields. This has been the mainstream treatment which (as we have discussed above) results in physically problematic solutions. The other approach is to introduce a background metric ansatz in terms of the Stückelberg scalars whose forms are also specially chosen. In this case by substituting the ansatz into the scalar field equations (which are derived by varying the gravitational potential coming from the mass terms) or by using the equivalent conservation equation of the mass contribution to the metric equation [17, 18] one obtains the particular scalar solutions and thus the accompanying background metric for these solutions. Then one is able to construct the associated cosmological equations whose solutions are under consideration. In defining the mass contribution to the metric sector, usually the effective energy-momentum tensor is introduced. In particular, when one considers the cosmological solutions this leads to an effective ideal fluid contribution to the cosmological dynamics apart from the matter presence. However, a great majority of the literature is restricted to only a

cosmological constant contribution via the effective fluid for the purpose of achieving self-accelerating solutions.

In the present paper instead of assigning a particular ansatz for the background metric in terms of the Stückelberg scalars (for example, like the case in Ref. [19]) our technical objective will be to derive its functional form constructed from the scalar and the physical metric solutions when it is diagonal with the entries that are functions of a single coordinate, so that the field equations are satisfied with such a choice of background metric. By focusing on the physical FLRW metrics we will be able to derive an extensive class of exact FLRW cosmological solutions of massive gravity for a functionally parametrized set of general background metrics for which the Stückelberg dependence is implicit. Therefore, in the following analysis our main concern will be to design the background metrics necessary to construct exact FLRW solutions of the theory. This approach is quite different than the ones used in other treatments (referred to above), which either used a flat background metric or introduced a particular Stückelberg dependence and solved the scalar and cosmological metric solutions thereafter. On the contrary in the following original solution construction technique which starts from an arbitrary background metric by considering the field equations which couple it to the scalar and the gravity sectors of the theory we will devise a method to find its functional form in terms of the Stückelberg-sector solutions, and the FLRW physical metric that is necessary to satisfy all the nonlinear field equations of the theory. Our methodology in this direction will be in parallel with that used in Ref. [23] (where a class of solutions of the minimal ghost-free massive gravity model [5, 6, 7] were constructed), and it will follow the same mathematical track as in Ref. [24]. Within this general approach one finds the means to decouple the scalar and gravity field equations. Essentially the decoupling follows an identification or a definition of the contribution coming from the Stückelberg scalar fields to the metric equation as an effective cosmological constant and an energy-momentum tensor which is well known in the literature. Since the covariant constancy of this contribution is equivalent to the scalar field equations from the above-mentioned identification, one can transform the scalar equations to a constraint condition on the effective energy-momentum tensor. Therefore, one may group all the dynamical equations of the theory in the metric sector in which an effective matter contribution is included. Such a compactified and scalar-free way of writing the metric equation results in the well-known Einstein form in the presence of some effective matter which obeys a conservation equation that is equivalent

lent to the dynamical equations of the scalars. Consequently in this respect one decouples the scalar and metric sectors by collecting all the dynamics in the metric sector and singling out an algebraic equation, which must be satisfied by the background and physical metrics as well as the scalars and the effective matter (which originally stems from the definition of the effective cosmological constant and the energy-momentum tensor contribution of the Stückelberg fields). Owing to its role in this decoupling mechanism of the field equations, we call this algebraic equation the solution ansatz in the following treatment. With this underlying general framework the main purpose of this paper will be to focus on the homogeneous and isotropic cosmological solutions of the theory. Hence, in this specialization apart from using the FLRW physical metric we will also consider an ideal-fluid form for the effective energy-momentum tensor of the solution ansatz. In this case, performing the above-mentioned decoupling will lead us to the standard cosmological dynamics whose well-known solutions depend on the equation of state of the ideal effective fluid (which in our derivation is completely arbitrary) to generate and parametrize the solutions. We will also derive the diagonal solutions of the algebraic ansatz equation for the background metric in terms of the energy density and the pressure of the effective fluid, the scale factor of the FLRW metric, and the Stückelberg scalar fields. In that regard we will be able to show that when one specifies the Stückelberg scalar fields arbitrarily and solves the effective energy density, the effective pressure, and the scale factor from the completely decoupled cosmological sector, one can construct the necessary background metrics which accompany these exact FLRW solutions of massive gravity when satisfying the field equations.

As we have remarked above, a similar analysis in Ref. [24] was used to derive another class of exact FLRW cosmological solutions of massive gravity. However, there a different and a more involved solution ansatz was at work which led to a nonstandard and a nonphysical conservation law for the effective ideal fluid arising via the application of the covariant constancy constraint on the ansatz equation (which is equivalent to the scalar field equations). Thus the resulting nonstandard continuity equation suggests that the effective ideal fluid which is a pseudo-ontological ingredient of the theory that takes the role of summarizing the collective effects of the mass degrees of freedom in the physical metric sector behaves (unlike the ordinary matter) nonphysically. The notion of nonphysicality here is due to the fact that such a conservation law cannot be consistent with the first law of thermodynamics, which would be the case for a fluid which exhibits a usual energy-momentum

conservation law. In the present paper on the other hand by staying in a similar solution scheme our main concern will be to construct completely new exact FLRW solutions of the theory which are generated by a somewhat simpler ansatz, which leads to solutions via physical-like effective fluids that have a standard energy-momentum conservation law that is compatible with the first law of thermodynamics.

2 The Cosmology of Physical Effective Fluids

In the following analysis our framework will be the general massive gravity action [5]

$$S = -M_p^2 \int \left[R * 1 - 2m^2 \sum_{n=0}^3 \beta_n e_n(\sqrt{\Sigma}) * 1 + \Lambda * 1 \right] - S_{matt}, \quad (2.1)$$

for generic coefficients β_n . The mass terms arise from the elementary symmetric polynomials

$$\begin{aligned} e_0 &\equiv e_0(\sqrt{\Sigma}) = 1, \\ e_1 &\equiv e_1(\sqrt{\Sigma}) = tr \sqrt{\Sigma}, \\ e_2 &\equiv e_2(\sqrt{\Sigma}) = \frac{1}{2}((tr \sqrt{\Sigma})^2 - tr(\sqrt{\Sigma})^2), \\ e_3 &\equiv e_3(\sqrt{\Sigma}) = \frac{1}{6}((tr \sqrt{\Sigma})^3 - 3 tr \sqrt{\Sigma} tr(\sqrt{\Sigma})^2 + 2 tr(\sqrt{\Sigma})^3) \end{aligned} \quad (2.2)$$

of the square root matrix $\sqrt{\Sigma}$ of

$$(\Sigma)^\mu{}_\nu = g^{\mu\rho} \partial_\rho \phi^a \partial_\nu \phi^b \bar{f}_{ab}, \quad (2.3)$$

which couples the inverse physical metric $g^{\mu\nu}$ to the fiducial one $\bar{f}_{ab}(\phi^c)$ via the four Stückelberg scalar fields $\{\phi^a(x^\mu)\}$. Now if one varies the action (2.1) with respect to the physical metric one obtains the metric equation [5]

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \Lambda g_{\mu\nu} + \frac{1}{2} m^2 \left[\sum_{n=0}^3 (-1)^n \beta_n \left(g_{\mu\lambda} Y_{n\nu}^\lambda(\sqrt{\Sigma}) \right. \right. \\ \left. \left. + g_{\nu\lambda} Y_{n\mu}^\lambda(\sqrt{\Sigma}) \right) \right] = G_N T_{\mu\nu}^{matt}, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned}
Y_0(\sqrt{\Sigma}) &= \mathbf{1}_4, \\
Y_1(\sqrt{\Sigma}) &= \sqrt{\Sigma} - \text{tr} \sqrt{\Sigma} \mathbf{1}_4, \\
Y_2(\sqrt{\Sigma}) &= (\sqrt{\Sigma})^2 - \text{tr} \sqrt{\Sigma} \sqrt{\Sigma} + \frac{1}{2}[(\text{tr} \sqrt{\Sigma})^2 - \text{tr}(\sqrt{\Sigma})^2] \mathbf{1}_4, \\
Y_3(\sqrt{\Sigma}) &= (\sqrt{\Sigma})^3 - \text{tr} \sqrt{\Sigma} (\sqrt{\Sigma})^2 + \frac{1}{2}[(\text{tr} \sqrt{\Sigma})^2 - \text{tr}(\sqrt{\Sigma})^2] \sqrt{\Sigma} \\
&\quad - \frac{1}{6}[(\text{tr} \sqrt{\Sigma})^3 - 3 \text{tr} \sqrt{\Sigma} \text{tr}(\sqrt{\Sigma})^2 + 2 \text{tr}(\sqrt{\Sigma})^3] \mathbf{1}_4,
\end{aligned} \tag{2.5}$$

where all terms are 4×4 matrices. On the other hand, the field equations of Eq.(2.1) for the Stückelberg scalar fields $\{\phi^a(x^\mu)\}$ can equivalently be written as

$$\nabla^\mu \left[\sum_{n=0}^3 (-1)^n \beta_n \left(g_{\mu\lambda} Y_{n\nu}^\lambda(\sqrt{\Sigma}) + g_{\nu\lambda} Y_{n\mu}^\lambda(\sqrt{\Sigma}) \right) \right] = 0. \tag{2.6}$$

Now let us consider the homogeneous and isotropic solutions of Eq.(2.4). For this reason we take the physical metric to be the FLRW one,

$$g = -dt^2 + \frac{a^2(t)}{1 - kr^2} dr^2 + a^2(t) r^2 d\theta^2 + a^2(t) r^2 \sin^2 \theta d\varphi^2, \tag{2.7}$$

and the physical matter as a perfect fluid whose energy-momentum tensor reads

$$T_{\mu\nu}^{matt} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}, \tag{2.8}$$

with $p = p(t)$ and $\rho = \rho(t)$ being the pressure and energy density of the fluid, respectively. We take the fluid four-velocity vector as $U_\mu = (1, 0, 0, 0)$ which is defined in the rest frame of the fluid. Aiming to derive the cosmological dynamics that admit solutions of Eq.(2.4) in the form (2.7), we propose the solution ansatz

$$\frac{1}{2} m^2 \left[\sum_{n=0}^3 (-1)^n \beta_n \left(g_{\mu\lambda} Y_{n\nu}^\lambda(\sqrt{\Sigma}) + g_{\nu\lambda} Y_{n\mu}^\lambda(\sqrt{\Sigma}) \right) \right] = C_1 m^2 g_{\mu\nu} + C_2 m^2 \tilde{T}_{\mu\nu}^{eff}, \tag{2.9}$$

where C_1 and C_2 are arbitrary constants and we assume that \tilde{T}^{eff} is also of the ideal fluid form,¹

$$\tilde{T}_{\mu\nu}^{eff} = (\tilde{\rho}(t) + \tilde{p}(t)) U_\mu U_\nu + \tilde{p}(t) g_{\mu\nu}. \tag{2.10}$$

¹For future convenience here we define the matrix $[\tilde{T}^{eff}]^\mu{}_\nu := \tilde{T}_{\mu\nu}^{eff}$.

At this point we should emphasize once more that both in Ref. [24] and here, following the introduction of an effective cosmological constant and a fluid in the solution ansatz the perspective is to derive the necessary background metric that will lead us to the solutions of the theory by solving the Stückelberg-sector fields and by constructing the cosmological dynamics ready to be solved for the scale factor, effective energy density, and pressure of the hypothetical fluid. In Ref. [24], however, the solutions were based on the assumption of an effective fluid that exhibits a nonphysical conservation law thus does not obey the first law of thermodynamics. In the present treatment on the other hand by being in a similar solution scheme as that in Ref. [24] we will be aiming to obtain completely new solutions of the theory which are generated by a physical-like effective fluid used in the ansatz which obeys the first law of thermodynamics. Therefore the solution ansatz differs from the one we employed in Ref. [24], whose unique form was dictated by the nonphysical nature of the corresponding effective fluid we considered. The nonphysical nature was due to the fact that the effective fluid Lagrangian in Ref. [24] was chosen in its most general form so that one does not have to use the first law of thermodynamics to derive the energy-momentum tensor (2.10) upon varying the Lagrangian with respect to the inverse metric. In this work we will consider the physically oriented effective fluids which obey the first law of thermodynamics,

$$d\tilde{\rho} = \tilde{\mu}d\tilde{n} + \tilde{n}\tilde{T}d\tilde{s}, \quad (2.11)$$

where $\tilde{\mu} = (\tilde{\rho} + \tilde{p})/\tilde{n}$, \tilde{n} , \tilde{T} , \tilde{s} are the chemical potential, the particle number density, the temperature, and the entropy per particle of the effective fluid respectively. If one defines the particle number flux density as

$$J^\mu = \tilde{n}\sqrt{-g}U^\mu, \quad (2.12)$$

and introduces the Lagrangian coordinate scalars α^A , as well as the Clebsch scalar potentials $\theta_1, \theta_2, \beta_A$ where $A = 1, 2, 3$, one can write the ideal fluid action [25] as

$$S_{IF} = \int dx^4 \sqrt{-g} \left[-\tilde{\rho} + \frac{1}{\sqrt{-g}} J^\mu (\partial_\mu \theta_1 + \tilde{s} \partial_\mu \theta_2 + \beta_A \partial_\mu \alpha^A) \right]. \quad (2.13)$$

Bearing in mind the equation of state $\tilde{\rho} = \tilde{\rho}(\tilde{n}, \tilde{s})$ and the fact that via Eq.(2.12) we have

$$\tilde{n} = \frac{\sqrt{-J^\mu J_\mu}}{\sqrt{-g}}, \quad (2.14)$$

the linearly independent fields composing the action functional (2.13) become $\{g^{\mu\nu}, J^\mu, \tilde{s}, \alpha^A, \theta_1, \theta_2, \beta_A\}$. By varying this action with respect to the inverse metric, using the field equations of J^μ [obtained by varying Eq.(2.13) with respect to J^μ], and demanding that the effective fluid satisfies the first law (2.11), via

$$\tilde{T}_{\mu\nu}^{eff} = -2 \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{IF})}{\delta(g^{\mu\nu})}, \quad (2.15)$$

one can derive the effective ideal fluid energy-momentum tensor (2.10). Hence, we observe that in order for the action (2.13) to yield Eq.(2.10) the fluid must be constrained to be a physical one that satisfies the first law of thermodynamics. One can furthermore show that if the field equations of the particle number flux densities J^μ are used in Eq.(2.13) then the Lagrangian of the physically behaving effective fluid becomes

$$\mathcal{L}_{IF} = \tilde{p}. \quad (2.16)$$

Putting this result aside, our next task is to identify the on-shell Lagrangian of the mass terms in Eq.(2.1), that is the Lagrangian-level ansatz which generates Eq.(2.9). From Eqs.(2.1) and (2.4) for the solutions which satisfy the ansatz (2.9), we can deduce that the variation with respect to the inverse physical metric must yield (on-shell)

$$\delta(2m^2\sqrt{-g}\sum_{n=0}^3\beta_n e_n) = -\sqrt{-g}[C_1 m^2 g_{\mu\nu} + C_2 m^2 \tilde{T}_{\mu\nu}^{eff}]\delta g^{\mu\nu}. \quad (2.17)$$

Therefore from our above discussion we conclude that for the solutions satisfying Eq.(2.9) we must have the on-shell identity

$$\sum_{n=0}^3 \beta_n e_n = C_1 + C_2 \tilde{p}. \quad (2.18)$$

We should remark once more that for this on-shell identity to hold we must demand that the effective ideal fluid whose energy-momentum tensor is used in the solution ansatz (2.9) must satisfy the first law of thermodynamics (2.11). For this reason it deserves to be called physical. Thus in order to be able to use the solution (2.9) and the corresponding Lagrangian ansatz (2.17) together we must introduce Eq.(2.11) as a mathematical constraint on the effective fluid. On the other hand to satisfy the Stückelberg scalar field

equations (2.6) we observe that our solution ansatz (2.9) demands that the equation

$$\nabla^\mu [C_1 g_{\mu\nu} + C_2 \tilde{T}_{\mu\nu}^{eff}] = 0, \quad (2.19)$$

must be satisfied by the solution-generating-and-parametrizing fields composing \tilde{T}^{eff} . By using the metric compatibility of the spin connection this constraint leads us to

$$\nabla^\mu \tilde{T}_{\mu\nu}^{eff} = 0, \quad (2.20)$$

which is in the form of the usual continuity or energy-momentum conservation equation for a physical fluid which we would expect to see following our assumption of a physical-like behavior for our solution-parametrizing effective fluid via demanding that it should satisfy Eq.(2.11). Hence unlike the case studied in Ref. [24], we get no modification to the fluid equation of the effective ideal fluid in this solution scheme. Our next task is to find the Stückelberg fields and the background metric in terms of the physical metric, the coefficients C_1, C_2 , and the pressure and the energy density of the effective fluid that satisfy the solution ansatz (2.9). Now, by using the matrix identity $g(\sqrt{\Sigma})^n = (g(\sqrt{\Sigma})^n)^T$, for any integer n [26], as well as the definitions of the elementary symmetric polynomials in Eq.(2.2), and the constraint (2.18) (which attributes a physical nature to the effective fluid introduced to parametrize the solution ansatz), from Eq.(2.9) (after some algebra) we obtain the algebraic matrix equation²

$$\begin{aligned} & -\beta_3(\sqrt{\Sigma})^3 + (\beta_2 + \beta_3 e_1)(\sqrt{\Sigma})^2 + (-\beta_1 - \beta_2 e_1 - \beta_3 e_2)(\sqrt{\Sigma}) \\ & + C_2 \tilde{p} \mathbf{1}_4 - C_2 g^{-1} \tilde{T}^{eff} = 0. \end{aligned} \quad (2.21)$$

If we take the trace of this equation by using Eqs.(2.2) and (2.18) we can write e_2 in terms of e_1 . It reads

$$e_2 = \frac{1}{\beta_2} [3C_1 - 3\beta_0 - 2\beta_1 e_1 + C_2 \tilde{T}^\mu_{\mu} - C_2 \tilde{p}], \quad (2.22)$$

where we define the contraction $\tilde{T}^\mu_{\mu} := g^{\mu\nu} \tilde{T}_{\mu\nu}^{eff}$. This relation reduces the number of elementary symmetric polynomial functions to one; namely, in our solution scheme e_1 remains to be specified by the particular solution chosen.

²The reader may refer to Ref. [24] for the details of a similar computation.

A subset of the general solutions of Eq.(2.21) with a different constant coefficient term were derived in Ref. [24] under the assumption of the diagonality of $\sqrt{\Sigma}$. If we adopt those solutions from Ref. [24] while bearing in mind the variant constant term in Eq.(2.21), then the background metric solutions can be given as

$$\bar{f} = diag\left(\frac{\mathcal{V}'_{00}}{(F_0(x^0))^2}, \frac{\mathcal{V}'_{11}}{(F_1(x^1))^2}, \frac{\mathcal{V}'_{22}}{(F_2(x^2))^2}, \frac{\mathcal{V}'_{33}}{(F_3(x^3))^2}\right), \quad (2.23)$$

where $\{F_\mu\}$ are arbitrary functions of only the single coordinate x^μ , and the definitions of the diagonal matrices \mathcal{V}' are given in the Appendix. On the other hand, the Stückelberg scalar fields that solve Eq.(2.21) become [23, 24]

$$\phi^c(x^c) = \pm \int F_c(x^c) dx^c. \quad (2.24)$$

Hence the reader may immediately realize that they are completely arbitrary. Having found the scalar-sector solutions arising from the ansatz (2.9), let us substitute Eq.(2.9) into the metric equation (2.4) to find the corresponding cosmological dynamics upon the choice of the FLRW metric (2.7). We get the equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \tilde{\Lambda}g_{\mu\nu} = G_N T_{\mu\nu}^{matt} - C_2 m^2 \tilde{T}_{\mu\nu}^{eff}, \quad (2.25)$$

where we have introduced the effective cosmological constant $\tilde{\Lambda} = \frac{1}{2}\Lambda - C_1 m^2$ which has a contribution coming from the ansatz (2.9). By referring to the ideal-fluid nature of the real and effective matter sources namely, by using the energy-momentum tensors (2.8) and (2.10), the computation of Eq.(2.25) for the metric (2.7) gives the t -component equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{G_N}{3}\rho - \frac{C_2 m^2}{3}\tilde{\rho} - \frac{\tilde{\Lambda}}{3}, \quad (2.26)$$

and the three identical spatial-component equations

$$\frac{2\ddot{a}}{a} = -\left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} - G_N p + C_2 m^2 \tilde{p} - \tilde{\Lambda}. \quad (2.27)$$

Now by using Eq.(2.26) in (2.27) we get the cosmic acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{G_N}{6}(3p + \rho) + \frac{C_2 m^2}{6}(3\tilde{p} + \tilde{\rho}) - \frac{\tilde{\Lambda}}{3}. \quad (2.28)$$

We realize that the Friedmann [Eq.(2.26)] and acceleration [Eq.(2.28)] equations are in the canonical form of the ordinary cosmological ones, and additional contributions to the cosmological constant, fluid pressure, and energy density come from the graviton mass sector terms in Eq.(2.1). The fluid equation

$$\dot{\rho} = -\frac{3\dot{a}}{a}(p + \rho), \quad (2.29)$$

of the real-matter ideal fluid (2.8) follows from its energy-momentum conservation law $\nabla^\mu T_{\mu\nu}^{matt} = 0$ which is computed for the FLRW metric choice (2.7). On the other hand a similar continuity equation

$$\dot{\tilde{\rho}} = -\frac{3\dot{a}}{a}(\tilde{p} + \tilde{\rho}), \quad (2.30)$$

which is also in the canonical form must be satisfied by the effective ideal fluid (which has physical-like properties) via the presence of the constraint equation (2.20) which has replaced the Stückelberg scalar field equations upon the introduction of the ansatz (2.9). Likewise in the standard cosmology case, Eqs.(2.29) and (2.30) must be solved together with the Friedmann equation to derive the cosmological solutions. We should remark here that as a result of our analysis we get a set of cosmological equations for massive gravity which do not get dynamical modification terms, i.e., they are in the same form as the equations of standard cosmology; hence, they exhibit the same solution moduli. The only extra contribution that appears due to the presence of the mass sector of the theory is the mathematical existence of an effective ideal fluid (which lacks any other physical role apart from the gravitational one, of course), in addition to the physical content of the Universe. Moreover, the equation of state of this effective ideal fluid is completely arbitrary.

3 Appendix

Here we will present the formal details of the solutions of Eq.(2.21) in parallel with the achievements of Ref. [24]. By assuming a diagonal form for g, Σ, \bar{f} , and \tilde{T}^{eff} we can give the three ($i = 1, 2, 3$) distinct solutions of the cubic equation (2.21) as

$$\sqrt{\Sigma} = \mathcal{V}_i, \quad (3.1)$$

where

$$\mathcal{V}_i = -\frac{1}{3}\mathbf{a}^{-1}[\mathbf{b}\mathbf{1}_4 + u_i\mathcal{U} + u_i^{-1}\mathcal{U}^{-1}(\mathbf{b}^2 - 3\mathbf{a}\mathbf{c})], \quad (3.2)$$

and $\mathbf{1}_4$ is the unit 4×4 matrix. The coefficients that differ for each solution are

$$u_1 = 1, \quad u_2 = \frac{1}{2}(-1 + i\sqrt{3}), \quad u_3 = \frac{1}{2}(-1 - i\sqrt{3}). \quad (3.3)$$

Also the matrix \mathcal{U} is

$$\mathcal{U} = \left[\frac{2\mathbf{b}^3 - 9\mathbf{a}\mathbf{b}\mathbf{c} + 27\mathbf{a}^2\mathbf{d} + \sqrt{(2\mathbf{b}^3 - 9\mathbf{a}\mathbf{b}\mathbf{c} + 27\mathbf{a}^2\mathbf{d})^2 - 4(\mathbf{b}^2 - 3\mathbf{a}\mathbf{c})^3}}{2} \right]^{1/3}. \quad (3.4)$$

Here the constant matrices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the same as those in Ref. [24]. They are

$$\begin{aligned} \mathbf{a} &= -\beta_3\mathbf{1}_4, \\ \mathbf{b} &= (\beta_2 + \beta_3 e_1)\mathbf{1}_4, \\ \mathbf{c} &= (-\beta_1 - \beta_2 e_1 - \beta_3 e_2)\mathbf{1}_4. \end{aligned} \quad (3.5)$$

However the matrix function \mathbf{d} differs from the functional form of the one in Ref. [24] it reads

$$\mathbf{d} = C_2\tilde{p}\mathbf{1}_4 - C_2g^{-1}\tilde{T}^{eff}. \quad (3.6)$$

In the rest frame of the effective fluid [via Eq.(2.7) and the effective ideal fluid energy-momentum tensor (2.10)], we can explicitly calculate \mathbf{d} , which becomes

$$\begin{aligned} \mathbf{d} &= C_2 \begin{pmatrix} \tilde{p} & 0 & 0 & 0 \\ 0 & \tilde{p} & 0 & 0 \\ 0 & 0 & \tilde{p} & 0 \\ 0 & 0 & 0 & \tilde{p} \end{pmatrix} - C_2 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & g_{11}^{-1} & 0 & 0 \\ 0 & 0 & g_{22}^{-1} & 0 \\ 0 & 0 & 0 & g_{33}^{-1} \end{pmatrix} \begin{pmatrix} \tilde{\rho} & 0 & 0 & 0 \\ 0 & \tilde{p}g_{11} & 0 & 0 \\ 0 & 0 & \tilde{p}g_{22} & 0 \\ 0 & 0 & 0 & \tilde{p}g_{33} \end{pmatrix} \\ &= \begin{pmatrix} C_2(\tilde{\rho} + \tilde{p}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.7)$$

Similarly, explicit matrix multiplication yields $\tilde{T}^\mu_\mu = 3\tilde{p} - \tilde{\rho}$, which appears in the relation (2.22) that can be used to eliminate e_2 in terms of e_1 in the

solutions (3.2). We should state at this point that within the solutions we have constructed, e_1 must be specified by the particular choice of solution of Eq.(2.21). In general, the solutions given in Eq.(3.2) can be complex; however, we have to consider the real ones. Now, if we concentrate on the real solutions in Eq.(3.2) by referring to the definition (2.3), Eq.(3.1) can be written as

$$\partial_\mu \phi^a \partial_\nu \phi^b \bar{f}_{ab} = \mathcal{V}'_{\mu\nu}, \quad (3.8)$$

where we have introduced

$$\mathcal{V}' = g\mathcal{V}^2 \quad (3.9)$$

and defined the tensor components

$$\mathcal{V}'_{\mu\nu} := [\mathcal{V}']^\mu{}_\nu. \quad (3.10)$$

One can show that [23, 24, 27] the background metric choice (2.23) and the completely arbitrary Stückelberg scalar field solutions (2.24) presented in Sec. II satisfy the set of first-order partial differential equations (3.8). Consequently, Eq.(2.24) forms a solution class of the Stückelberg sector of the massive gravity theory when the background metric is chosen as Eq.(2.23), whose explicit form depends on the specification of the effective energy density and the pressure via the cosmological dynamics. Therefore, in order to generate the cosmological solutions of the theory under inspection it is sufficient to solve Eqs.(2.26), (2.27), (2.29), and (2.30) for the scale factor, the effective and physical energy densities, and the pressure after specifying an equation of state for the effective and physical ideal fluids. Finally, we should emphasize one crucial fact here, namely, that the functional form of the matrix \mathcal{V}' is different than that in Ref. [24]. This is a consequence of the different solution ansatz used in the present work. The cosmological solutions (the background and physical metrics as well as the Stückelberg scalars) stemming from this ansatz are new, both in their mathematical notion and their physical content which we comment on in the Conclusion.

4 Conclusion

We have adopted the solution method developed in Refs. [23, 24] to derive new cosmological solutions of the ghost-free massive gravity theory. For an unspecified background metric, and a FLRW physical metric scenario, we first decoupled the Stückelberg scalars from the metric sector of the theory via a

specially chosen ansatz which introduces an effective cosmological constant and a fluid contribution of the mass sector to the metric equation. Then, we found solutions to the scalars and derived the background metric in terms of these scalar solutions, the physical FLRW metric, and the properties of the effective fluid so that they algebraically satisfy the solution ansatz. Later, we constructed the cosmological equations where the collective effect of the Stückelberg scalars and the background metric enter into the equations as an effective cosmological constant and a source in the form of the energy density and the pressure of a hypothetical effective ideal fluid. Therefore, after specifying a freely chosen equation of state for the effective fluid one may first solve the scale factor of the FLRW metric, the energy density, and the pressure of the effective fluid and then use these to construct the necessary background metric for which these solutions exist. Consequently, we have obtained a large class of cosmological solutions of the theory for diagonal-background metrics, where the diagonality has been the key ingredient to decouple the components of the ansatz equation (which is a matrix equation). The derived solutions compose a wide class because one has solution-parametrizing functional degrees of freedom in assigning a generic equation of state to the effective fluid. In addition, the solutions are also parametrized by two free coefficients. We should declare a fundamental difference between the outcome of the present work and that in Ref. [24], both of which use the same solution systematics. First of all, the solution ansatz and the resulting solutions are quite different for these two works. The basic reason for this lies in the nature of the ansatz-generating effective fluid. As we have mentioned before, the solutions obtained in Ref. [24] were parametrized by an effective ideal fluid which has a modified energy-momentum conservation law. This fact has structural influences not only on the solution ansatz, but on the cosmological dynamics as well. All the cosmological equations are modified due to the nonphysical nature of the effective fluid in Ref. [24]. On the other hand, in the present work where we found a new class of solutions of the theory the effective fluid is assumed to obey a physical energy-momentum conservation equation. This is reflected in two places: first, in the simplicity of the solution ansatz, and second, in the unmodified canonical form of the Friedmann equation and the continuity equations of the cosmological dynamics. We believe that the second of these is quite important. Fundamentally, reviving the canonical form of the cosmological dynamics as solutions within the context of massive gravity with just the addition of some hypothetical matter is the same as restoring Einsteinian cosmology in massive gravity. Hence,

the formalism and the results of the present work enable us to construct an inclusion map between the homogeneous and isotropic solutions of Einstein gravity and those in ghost-free massive gravity. One lacks such a connection for the other branch of the solution moduli constructed in Ref. [24]. This is not only an essential mathematical achievement but it also promises physically eligible solutions of massive gravity that may hold a cure for the dark energy problem of standard cosmology in the effective-matter domain. More specifically, the ordinary Einstein form of the cosmological dynamics enables one to combine the physical and effective-matter terms. This may help one to construct solutions in which the geometry of the solutions does not match with the required physical matter content (from an Einstein point of view), where the effective matter takes the role of compensation. We believe that such a significant result is the most outstanding contribution of the present work and deserves to be reported in its own right.

We should furthermore note that unlike the general achievements in the literature the cosmological solutions derived here cover a generic effective energy-momentum contribution in the cosmological dynamics rather than just a cosmological constant one. Moreover, the Stückelberg field solutions are not restricted to a particular functional set of solutions, as was the case in other similar works. Rather they are completely arbitrary in the solution framework we have addressed. As an example, the reader may compare the method applied here with the one used in Ref. [19], where it was proven that self-accelerating solutions exist for an isotropic background metric whose functional dependence on coordinates differs from the single-coordinate dependence assumed in this work. One realizes that therein, for a special choice of the background metric ansatz in terms of the Stückelberg fields, it has been shown that for a particular configuration of solutions that fixes the scalar fields (which depend on the scale factor unlike our case, also without exactly being solved) thus the background metric the effective contribution in the metric sector becomes a cosmological constant hence giving rise to self-accelerating cosmological solutions on the physical metric side. Such a comparison may reveal the extensive generality of the solution domain obtained here not only for the scalars and background metrics but also for the cosmological scenarios, due to the richness of their effective source feed originating from the large solution moduli of mass degrees of freedom. The reader should also appreciate that even trivial choices of Stückelberg field solutions are possible in this solution-generating method. The fundamental reason for this extension of freedom is that we are able to design the background metric

for a desired set of scalar and physical metric solutions. On the other hand, the price we pay for this is the nonstandard and complicated form of the background metric.

Similar to the structure of the solutions in Ref. [24] the freedom of choice of the equation of state of the effective matter enables us to construct a rich class of new cosmological-solution scenarios of the ghost-free massive gravity, which are certainly distinct from the ones obtained in Ref. [24] both in their mathematical structures and their physical nature. In particular, this broad class of solutions contains the self-accelerating ones. For example, as a special case all the solutions with $C_2 = 0$ are self-accelerating as the overall contribution to the metric sector in this case becomes an effective cosmological constant. We leave a detailed study and classification of these solutions to a later work. We should also note that the general application domain of the solution method of this work is not restricted to the cosmological cases only and it can be used to derive other solutions of the theory. Besides even though we have assumed diagonality and a special form for the background metric, one may work out the general solutions of the ansatz equation without assuming a particular form. This would result in a system of coupled first-order partial differential equations but in this case the important achievement would be to obtain the most general set of FLRW solutions of massive gravity. Finally, we point out that a future direction could be to inspect how to accommodate the class of solutions presented here in the context of bigravity.

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